Anosov Closing Lemma

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Set-up

Let M be a smooth manifold, $U \subset M$ an open subset, $f : U \to M$ a C^1 diffeomorphism onto its image, and $\Lambda \subset U$ a compact f-invariant set, i.e., $f\Lambda \subset \Lambda$.

Definition

The set Λ is called a **hyperbolic set for the map** f if there exists a Riemannian metric in an open neighborhood U of Λ and $\lambda < 1 < \mu$ such that for any point $x \in \Lambda$ the sequence of differentials $(Df)_{f_x^n} : T_{f_x^n} M \to T_{f_x^{n+1}} M, n \in \mathbb{Z}$, admits a (λ, μ)-splitting, i.e., there exist decompositions $T_{f_x^n} M = E_n^s \oplus E_n^u$ such that $(Df)_{f_x^n} E_n^{s/u} = E_{n+1}^{s/u}$ and

$$\left\| (Df)_{f_x^n} \big|_{E_n^s} \right\| \leq \lambda, \quad \left\| (Df)_{f_x^n}^{-1} \big|_{E_{n+1}^u} \right\| \leq \mu^{-1}.$$

Example. The Arnold's cat map on 2-torus: $f = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) : \mathbb{T}^2 \to \mathbb{T}^2$, $\lambda = \frac{3-\sqrt{5}}{2}$, $\mu = \frac{3+\sqrt{5}}{2}$.

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Definition

We call a sequence $x_0, x_1, \ldots, x_{m-1}, x_m = x_0$ of points a **periodic ϵ-orbit** if $\text{dist}(fx_k, x_{k+1}) < \epsilon$ for $k = 0, \ldots, m-1$.

Theorem

Let Λ be a hyperbolic set for $f : U \to M$. Then there exists an open neighborhood $V \supset \Lambda$ and $C, \epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ and any periodic ϵ-orbit $(x_0, \ldots, x_m) \subset V$ there is a point $y \in U$ such that $f^m y = y$ and dist$(f^k y, x_k) < C\epsilon$ for $k = 0, \ldots, m-1$.

Reference: Introduction to the Mordern Theory of Dynamical Systems, Katok & Hasselblatt.